Q). max. $Z = 10x_1 - x_1^2 + 10x_2 - x_2^2$

S.t.

$$x_1 + x_2 \leq 9$$

$$x_1 - x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Solution:

Since NLPP is maximizing, all the constraints should have "$\leq$" sign in its standard form.

So, let us rewrite the NLPP as follows

max. $Z = 10x_1 - x_1^2 + 10x_2 - x_2^2$

S.t.

$$x_1 + x_2 \leq 9$$

$$-x_1 + x_2 \leq -6$$

$$x_1, x_2 \geq 0$$

$\therefore$ The Hessian matrix is

$$H^B = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

The principal minors are

$\Delta_1 = -2$

$\Delta_2 = 4$

which are of the alternate sign.

$\Rightarrow f(X)$ is concave function of X

Also,

$h_1(X) = x_1 + x_2 - 9 \leq 0$

$h_2(X) = -x_1 + x_2 + 6 \leq 0$

are convex functions of X.

$\therefore$ The Kuhn-Tucker necessary conditions are also the sufficient condition for existence of maximum value of $f(X)$.

The N/c for existence of maxima of $f(X)$.

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{2} \lambda_i \frac{\partial h_i}{\partial x_j} = 0$$

$\lambda_i h_i(X) = 0$

$h_i(X) \leq 0$

and $\lambda_i \geq 0$; $i = 1, 2$

$$\Rightarrow \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$$

and $$\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$$

$$\lambda_1 h_1(x) = 0$$

$$\lambda_2 h_2(x) = 0$$

$$h_1(x) \leq 0$$

$$h_2(x) \leq 0$$

$$\lambda_1, \lambda_2 \geq 0.$$

$$\Rightarrow 10 - 2x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{——— (1)}$$

$$10 - 2x_2 - \lambda_1 - \lambda_2 = 0 \quad \text{——— (2)}$$

$$\lambda_1 (x_1 + x_2 - 9) = 0 \quad \text{——— (3)}$$

$$\lambda_2 (-x_1 + x_2 + 6) = 0 \quad \text{——— (4)}$$

$$x_1 + x_2 - 9 \leq 0 \quad \text{——— (5)}$$

$$-x_1 + x_2 + 6 \leq 0 \quad \text{——— (6)}$$

$$\lambda_1, \lambda_2 \geq 0 \quad \text{——— (7)}$$

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Case I

when $\lambda_1 = 0 = \lambda_2$

① $\Rightarrow x_1 = 5$

② $\Rightarrow x_2 = 5$

clearly, $x_1 = x_2 = 5$ does not satisfy eq^n ⑤ and ⑥

$\therefore$ this solution is inadmissible

$\Rightarrow \lambda_1 \neq 0 \neq \lambda_2$

Case II:

when $\lambda_1 \neq 0$ and $\lambda_2 = 0$

① $\Rightarrow 10 - 2x_1 - \lambda_1 = 0$

$\Rightarrow x_1 = \frac{1}{2}(10 - \lambda_1)$

② $\Rightarrow 10 - 2x_2 - \lambda_1 = 0$

$\Rightarrow x_2 = \frac{1}{2}(10 - \lambda_1)$

since $\lambda_1 \neq 0$

③ $\Rightarrow x_1 + x_2 - 9 = 0$

or, $\frac{1}{2}(10-\lambda_1) + \frac{1}{2}(10-\lambda_1) - 9 = 0$

or, $10 - \lambda_1 + 10 - \lambda_1 - 18 = 0$

or, $2 - 2\lambda_1 = 0$

$\therefore \boxed{\lambda_1 = 1}$

$\therefore \quad x_1 = \frac{9}{2}$

and $x_2 = \frac{9}{2}$

which does not satisfy eq^n 6

$\therefore$ this solution is also inadmissible

<u>Case III</u>:

when $\lambda_1 = 0, \lambda_2 \neq 0$

1 $\Rightarrow 10 - 2x_1 + \lambda_2 = 0$

$\Rightarrow x_1 = \frac{1}{2}(10 + \lambda_2)$

2 $\Rightarrow 10 - 2x_2 - \lambda_2 = 0$

$\Rightarrow x_2 = \frac{1}{2}(10 - \lambda_2)$

since $\lambda_2 \neq 0$

④ $\Rightarrow -x_1 + x_2 + 6 = 0$

or, $-\frac{1}{2}(10+\lambda_2) + \frac{1}{2}(10-\lambda_2) + 6 = 0$

or, $-(10+\lambda_2) + (10-\lambda_2) + 12 = 0$

or, $-2\lambda_2 + 12 = 0$

$$\boxed{\lambda_2 = 6}$$

$\therefore x_1 = \frac{1}{2}(10+6) \Rightarrow \boxed{x_1 = 8}$

$x_2 = \frac{1}{2}(10-6) \Rightarrow \boxed{x_2 = 2}$

which does not satisfy eq^n ⑤.
Hence this solution is also not acceptable

Case - IV

when $\lambda_1 \neq 0 \neq \lambda_2$

eq^n ① $\Rightarrow x_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + 10)$

eq^n ② $\Rightarrow x_2 = \frac{1}{2}(-\lambda_1 - \lambda_2 + 10)$

since, $\lambda_1 \neq 0$

3 $\Rightarrow$ $x_1 + x_2 - 9 = 0$

or, $\frac{1}{2}(-\lambda_1 + \lambda_2 + 10) + \frac{1}{2}(-\lambda_1 - \lambda_2 + 10) - 9 = 0$

or, $-\lambda_1 + \lambda_2 + 10 - \lambda_1 - \lambda_2 + 10 - 18 = 0$

or, $-2\lambda_1 + 2 = 0$

$\Rightarrow$ $\boxed{\lambda_1 = 1}$

since, $\lambda_2 \neq 0$

4 $\Rightarrow$ $-x_1 + x_2 + 6 = 0$

or, $-\frac{1}{2}(-\lambda_1 + \lambda_2 + 10) + \frac{1}{2}(-\lambda_1 - \lambda_2 + 10) + 6 = 0$

or, $\lambda_1 - \lambda_2 - 10 - \lambda_1 - \lambda_2 + 10 + 12 = 0$

or, $-2\lambda_2 + 12 = 0$

$\Rightarrow$ $\boxed{\lambda_2 = 6}$

$\therefore$ $x_1 = \frac{1}{2}(-1 + 6 + 10) \Rightarrow \boxed{x_1 = \frac{15}{2}}$

$x_2 = \frac{1}{2}(-1 - 6 + 10) \Rightarrow \boxed{x_2 = \frac{3}{2}}$

which satisfy eq^n (5) and (6) and Hence are the required solution of NLPP

$$\max. Z = f(x)$$

$$= 10x_1 - x_1^2 + 10x_2 - x_2^2$$

$$= 10\left(\frac{15}{2}\right) - \left(\frac{15}{2}\right)^2 + 10\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2$$

$$= \frac{63}{2}$$

$\therefore$ the optimal sol^n is

$$\left.\begin{array}{l} \max. Z = \frac{63}{2} \\ x_1 = \frac{15}{2}, \quad x_2 = \frac{3}{2} \end{array}\right\} \text{Ans.}$$